HIGHER SPIN SYMMETRIES, STAR-PRODUCT AND RELATIVISTIC EQUATIONS IN ADS SPACE

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Abstract

We discuss general properties of the theory of higher spin gauge fields in AdS_4 focusing on the relationship between the star-product origin of the higher spin symmetries, AdS geometry and the concept of space-time locality. A full list of conserved higher spin currents in the flat space of arbitrary dimension is presented.

Based on the talks given at QFTHEP'99 (Moscow, May 1999), Supersymmetries and Quantum Symmetries (Dubna, July 1999), Bogolyubov Conference Problems of Theoretical and Mathematical Physics (Moscow-Dubna-Kyiv, Sept-Oct. 1999) and Symmetry and Integrability in Mathematical and Theoretical Physics, the Conference in Memory of M.V.Saveliev (Protvino, January 2000)

1   Introduction

A theory of fundamental interactions is presently identified with still mysterious M-theory [1] which is supposed to be some relativistic theory having d=11 SUGRA as its low-energy limit. M-theory gives rise to superstring models in $d \leq 10$ and provides a geometric explanation of dualities. A particularly interesting version of the M-theory is expected to have anti-de Sitter (AdS) geometry explaining duality between AdS SUGRA and conformal models at the boundary of the AdS space [2]. More recently it has been realized that the star-product (Moyal bracket) plays important role in a certain phase of M-theory with nonvanishing vacuum expectation value of the antisymmetric field $B_{\underline{nm}}$ [3, 4]. In the limit $\alpha' \to 0$, $B_{\underline{nm}} = const$ string theory reduces to the noncommutative Yang-Mills theory [5].

The most intriguing question is: "what is M-theory?". It is instructive to analyze the situation from the perspective of the spectrum of elementary excitations. Superstrings describe massless modes of lower spins $s \leq 2$ like graviton ($s = 2$), gravitino ($s = 3/2$), vector bosons ($s = 1$) and matter fields with spins 1 and 1/2, as well as certain antisymmetric tensors. On the top of that there is an infinite tower of massive excitations of all spins. Since the corresponding massive parameter is supposed to be large, massive higher spin (HS) excitations are not directly observed at low energies. They are important however for the consistency of the theory. Assuming that M-theory is some relativistic theory that admits a covariant perturbative interpretation we conclude that it should contain HS modes to describe superstring models as its particular vacua. There are two basic alternatives: (i) $m \neq 0$: HS modes in M-theory are massive or (ii) $m = 0$: HS modes in M-theory are massless while massive HS modes in the superstring models result from compactification of extra dimensions.

Each of these alternatives is not straightforward. For the massive option, no consistent superstring theory is known beyond ten dimensions and therefore there is no good guiding principle towards M-theory from that side. For the massless case the situation is a sort of opposite: there is a very good guiding principle but it looks like it might be too strong. Indeed, massless fields of high spins are gauge fields [6]. Therefore this type of theories should be based on some HS gauge symmetry principle with the symmetry generators corresponding to various representations of the Lorentz group. It is however a hard problem to build a nontrivial theory with HS gauge symmetries. One argument is due to the Coleman-Mandula theorem and its generalizations [7] which claim that symmetries of S-matrix in a non-trivial (i.e., interacting) field theory in a flat space can only have sufficiently low spins. Direct arguments come [8] from the explicit attempts to construct HS gauge interactions in the physically interesting situations (e.g. when the gravitational interaction is included).

However, some positive results [9] were obtained on the existence of consistent interactions of HS gauge fields in the flat space with the matter fields and with themselves but not with gravity. Somewhat later it was realized [10] that the situation changes drastically once, instead of the flat space, the problem is analyzed in the AdS space with nonzero curvature Λ. This generalization led to the solution of the problem of consistent HS-gravitational interactions in the cubic order at the action level [10] and in all orders in interactions at the level of equations of motion [11]. The role of AdS background in HS gauge theories is very important. First it cancels the Coleman-Mandula argument which is hard to implement in the AdS background.

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From the technical side the cosmological constant allows new types of interactions with higher derivatives, which have a structure $\Delta S^{int}_{p,n,m,k} \sim \Lambda^p \partial^n \phi \partial^m \phi \partial^k \phi$, where ϕ denotes any of the fields involved and p can take negative powers to compensate extra dimensions carried by higher derivatives of the fields in the interactions (an order of derivatives which appear in the cubic interactions increases linearly with spin [9, 10]). An important general conclusion is that Λ should necessarily be nonzero in the phase with unbroken HS gauge symmetries. In that respect HS gauge theories are analogous to gauged supergravities with charged gravitinos which also require $\Lambda \neq 0$ [12].

HS gauge theories contain infinite sets of spins $0 \leq s < \infty$. This implies that HS symmetries are infinite-dimensional. If HS gauge symmetries are spontaneously broken by one or another mechanism, then, starting from the phase with massless HS gauge fields, one will end up with a spontaneously broken phase with all fields massive except for a subset corresponding to an unbroken subalgebra. (The same time a value of the cosmological constant will be redefined because the fields acquiring a nonvanishing vacuum expectation value may contribute to the vacuum energy.) A most natural mechanism for spontaneous breakdown of HS gauge symmetries is via dimensional compactification. It is important that in the known d=3 and d=4 examples the maximal finite-dimensional subalgebras of the HS superalgebras coincide with the ordinary AdS SUSY superalgebras giving rise to gauged SUGRA models. Provided that the same happens in higher dimensional models, this opens a natural way for obtaining superstring type theories in $d \leq 10$ starting from some maximally symmetric HS gauge theory in $d \geq 11$.

2 Higher Spin Currents

Usual inner symmetries are related via the Noether theorem to the conserved spin 1 current that can be constructed from different matter fields. For example, a current constructed from scalar fields in an appropriate representation of the gauge group

$$J^{\underline{n}}{}_i{}^j = \bar{\phi}_i \partial^{\underline{n}} \phi^j - \partial^{\underline{n}} \bar{\phi}_i \phi^j \tag{1}$$

is conserved on the solutions of the scalar field equations

$$\partial_{\underline{n}} J^{\underline{n}j}_i = \bar{\phi}_i (\Box + m^2)\phi^j - (\Box + m^2)\bar{\phi}_i \phi^j . \tag{2}$$

(Underlined indices are used for differential forms and vector fields in d-dimensional space-time, i.e. $\underline{n} = 0, \ldots, d-1$, while i and j are inner indices.)

Translational symmetry is associated with the spin 2 current called stress tensor. For scalar matter it has the form

$$T^{\underline{mn}} = \partial^{\underline{m}} \phi \partial^{\underline{n}} \phi - \frac{1}{2}\eta^{\underline{mn}} \left(\partial_{\underline{r}} \phi \partial^{\underline{r}} \phi - m^2 \phi^2 \right) . \tag{3}$$

Supersymmetry is based on the conserved current called supercurrent. It has fermionic statistics and is constructed from bosons and fermions. For massless scalar ϕ and massless spinor ψ_ν it has the form

$$J^{\underline{n}}{}_\nu = \partial_{\underline{m}} \phi (\gamma^{\underline{m}} \gamma^{\underline{n}} \psi)_\nu , \tag{4}$$

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where $\gamma^n{}_\mu{}^\nu$ are Dirac matrices in d dimensions. Non-underlined indices $m, n, \ldots = 0 \div d-1$ are treated as vector indices in the fiber. (The difference between underlined and non-underlined indices is irrelevant in the flat space).

The conserved charges, associated with these conserved currents, correspond, respectively, to generators of inner symmetries $T^i{}_j$, space-time translations P^n and supertransformations Q^i_ν. The conserved current associated with Lorentz rotations can be constructed from the symmetric stress tensor

$$S^{\underline{n}\,\underline{ml}}_{;} = T^{\underline{nm}}\,x^{\underline{l}} - T^{\underline{nl}}\,x^{\underline{m}}, \qquad T^{\underline{nm}} = T^{\underline{mn}}. \tag{5}$$

These exhaust the standard lower spin conserved currents usually used in the field theory.

The list of lower spin currents admits a natural extension to HS currents containing higher derivatives of the physical fields. The HS currents associated with the integer spin s

$$J^{\underline{n}}_{;m_1\ldots m_t, n_1\ldots n_{s-1}} \tag{6}$$

are vector fields (index $\underline{n}$) taking values in all representations of the Lorentz group described by the traceless two-row Young diagrams

$$\tag{7}$$

with $0 \le t \le s-1$. This means that the currents $J^{\underline{n};m_1\ldots m_t, n_1\ldots n_{s-1}}$ are symmetric both in the indices n and m, satisfy the relations

$$(s-1)(s-2)J^{\underline{n}}_{;m_1\ldots m_t,{}^r{}_{n_3\ldots n_{s-1}}} = 0, \tag{8}$$

$$t(s-1)J^{\underline{n}}_{;rm_2\ldots m_t,{}^r{}_{n_2\ldots n_{s-1}}}, \qquad t(t-1)J^{\underline{n}}_{;m_3\ldots m_t r}{}^r{}_{,n_1\ldots n_{s-1}} = 0, \tag{9}$$

and obey the antisymmetry property

$$J^{\underline{n}}_{;m_2\ldots m_t\{n_s,n_1\ldots n_{s-1}\}_n} = 0, \tag{10}$$

implying that symetrization over any s indices n and/or m gives zero.

Let us now explain notation, which simplifies analysis of complicated tensor structures and is useful in the component analysis. Following [13] we combine the Einstein rule that upper and lower indices denoted by the same letter are to be contracted with the convention that upper (lower) indices denoted by the same letter imply symmetrization which should be carried out prior contractions. With this notation it is enough to put a number of symmetrized indices in brackets writing e.g. $X_{n(p)}$ instead of $X_{n_1\ldots n_p}$.

Now, the HS currents are $J^{\underline{n}}_{;m(t),n(s-1)}$ $(1 \le t \le s-1)$ while the conditions (8)-(10) take the form

$$J^{\underline{n}}_{;m(t),n(s-2)}{}^n = 0, \qquad J^{\underline{n}}_{;m(t-1)}{}^n{}_{,n(s-1)} = 0, \qquad J^{\underline{n}}_{;m(t)}{}^m{}_{,n(s-1)} = 0, \tag{11}$$

and

$$J^{\underline{n}}_{;m(t-1)n,n(s-1)} = 0 \,. \tag{12}$$

The HS supercurrents associated with half-integer spins

$$J^{\underline{n}}_{;m_1\ldots m_t,n_1\ldots n_{s-3/2};\nu} \tag{13}$$

are vector fields (index $\underline{n}$) taking values in all representations of the Lorentz group described by the γ -transversal two-row Young diagrams

$$\tag{14}$$

i.e., the irreducibility conditions for the HS supercurrents $J^{\underline{n}}_{;m(t),n(s-3/2);\nu}$ read

$$t J^{\underline{n}}_{;m(t-1)n,n(s-3/2);\nu} = 0 \tag{15}$$

and

$$(s - 3/2)\,\gamma^n{}_\mu{}^\nu J^{\underline{n}}_{;m(t),n(s-3/2);\nu} = 0 \,. \tag{16}$$

From these conditions it follows that

$$(s - 3/2)\gamma^m{}_\mu{}^\nu J^{\underline{n}}_{;m(t),n(s-3/2);\nu} = 0 \tag{17}$$

and all tracelessness conditions (8) and (9) are satisfied.

To avoid complications resulting from the projection to the space of irreducible (i.e. traceless or $\gamma-$ transversal) two–row Young diagrams we study the currents

$$J^{\underline{n}}(\xi) = J^{\underline{n}}_{;}{}^{m(t)\,,\,n(s-1)}\xi_{m(t),n(s-1)}, \quad J^{\underline{n}}(\xi) = \xi_{m(t),n(s-3/2);}{}^\nu J^{\underline{n}}_{;}{}^{m(t)\,,\,n(s-3/2)}{}_{;\nu}, \tag{18}$$

where $\xi_{m(t),n(s-1)}$ and $\xi_{m(t),n(s-3/2);}{}^\nu$ are some constant parameters which themselves satisfy analogous irreducibility conditions. The conservation law then reads

$$\partial_{\underline{n}} J^{\underline{n}}(\xi) = 0 \,. \tag{19}$$

The currents corresponding to one-row Young diagrams (i.e. those with $t = 0$) generalize the spin 1 current (1), supercurrent (4) and stress tensor (3). An important fact is that they can be chosen in the form

$$J^m{}_{;,}{}^{n(s-1)}\xi_{n(s-1)} = T^{m\,n(s-1)}\xi_{n(s-1)} \tag{20}$$

$$J^m{}_{;,}{}^{n(s-3/2)}{}_{;\nu}\xi_{n(s-3/2);}{}^\nu = T^{m\,n(s-3/2)}{}_{;\nu}\xi_{n(s-3/2);}{}^\nu \,, \tag{21}$$

with totally symmetric conserved currents $T^{n(s)}$ or supercurrents $T^{n(s-1/2)}{}_{;\nu}$,

$$\xi_{n(s-1)}\partial_n T^{n(s)} = 0 \,, \qquad \xi_{n(s-3/2);}{}^\nu \partial_n T^{n(s-1/2)}{}_{;\nu} = 0 \tag{22}$$

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$(\xi^n{}_{n(s-2)} = 0, \ (\xi_{n(s-3/2)}\gamma^n)^\nu = 0)$.

Analogously to the formula (5) for the angular momenta current, the symmetric (super)currents T allow one to construct explicitly x-dependent HS "angular" currents. An observation is that the angular HS (super)currents

$$J^{\underline{n}}(\xi) = T^{\underline{n}n(s-1)}x^{m(t)}\xi_{m(t),n(s-1)}, \quad J^{\underline{n}}(\xi) = T^{\underline{n}\,n(s-3/2)}{}_{;\nu}x^{m(t)}\xi_{m(t),n(s-3/2);}{}^\nu, \tag{23}$$

where we use the shorthand notation

$$x^{m(s)} = \underbrace{x^m \ldots x^m}_{s}, \tag{24}$$

also conserve as a consequence of (22) because when the derivative in (19) hits a factor of x^m, the result vanishes by symmetrization of too many indices in the parameters ξ forming the two-row Young diagrams.

Since the parameters $\xi_{m(t),n(s-1)}$ and $\xi_{m(t),n(s-3/2);}{}^\nu$ are traceless and $\gamma-$transversal, only the double traceless part of $T^{\underline{n}n(s-1)}$

$$T^{n(2)}{}_{n(s-2)} = 0, \qquad s \geq 4 \tag{25}$$

and triple $\gamma-$transversal part of $T^{\underline{n}\,n(s-3/2)}{}_{;\nu}$

$$\gamma^n T^n{}_{n(s-3/2)} = 0, \qquad s \geq 7/2. \tag{26}$$

contribute to (23). These are the (super)currents of the formalism of symmetric tensors (tensor-spinors) [6, 14]. The currents with integer spins $T^{n(s)}$ were considered in [15, 16] for the particular case of massless matter fields.

Integer spin currents built from scalars of equal masses

$$(\Box + m^2)\phi^i = 0 \tag{27}$$

have the form

$$\begin{aligned} T^{n(2k)ij} = & \ (\partial^{n(k)}\phi^i\partial^{n(k)}\phi^j - \frac{k}{2}\eta^{nn}\partial^{n(k-1)}\partial_m\phi^i\partial^{n(k-1)}\partial^m\phi^j \\ & + \frac{k}{2}m^2\eta^{nn}\partial^{n(k-1)}\phi^i\partial^{n(k-1)}\phi^j) + i \leftrightarrow j \end{aligned} \tag{28}$$

for even spins and

$$T^{n(2k+1)ij} = (\partial^{n(k+1)}\phi^i\partial^{n(k)}\phi^j) - i \leftrightarrow j \tag{29}$$

for odd spins, where we use notation analogous to (24)

$$\partial^{n(s)} = \underbrace{\partial^n \ldots \partial^n}_{s}. \tag{30}$$

HS supercurrents built from scalar and spinor with equal masses

$$(\Box + m^2)\phi = 0, \qquad (i\partial_{\underline{n}}\gamma^{\underline{n}} + m)\psi_\nu = 0 \tag{31}$$

read

$$T^{n(k+1)}{}_{;\nu} = \partial^{n(k+1)}\phi\psi_\nu - \frac{k+1}{2}\Big((\gamma^n\gamma^l\psi)_\nu\partial^{n(k)}\partial_l\phi + im(\gamma^n\psi)_\nu\partial^{n(k)}\phi\Big). \tag{32}$$

Inserting these expressions into (23) we obtain the set of conserved "angular" HS currents of even, odd and half-integer spins. The usual angular momentum current corresponds to the case $s = 2$, $t = 1$.

Remarkably, the conserved HS currents listed above are in one–to–one correspondence with the HS gauge fields (1-forms) $\omega_{\underline{n};m(t),n(s-1)}$ and $\omega_{\underline{n};m(t),n(s-3/2);\nu}$ introduced for the boson [17] and fermion [18] cases in arbitrary d. To the best of our knowledge, the fact that any of the HS gauge fields has a dual conserved current is new. Of course, such a correspondence is expected because, like the gauge fields of the supergravitational multiplets, the HS gauge fields should take their values in a (infinite-dimensional) HS algebra identified with the global symmetry algebra in the corresponding dynamical system (this fact is explicitly demonstrated below for the case of $d = 4$). The HS currents can then be derived via the Noether theorem from the global HS symmetry and give rise to the conserved charges identified with the Hamiltonian generators of the same symmetries.

A few comments are now in order.

HS currents contain higher derivatives. Therefore, HS symmetries imply, via the Noether procedure, the appearance of higher derivatives in interactions. The immediate question is whether HS gauge theories are local or not. As we shall see the answer is "yes" at the linearized level and "probably not" at the interaction level.

Nontrivial (interacting) theories exhibiting HS symmetries are formulated in AdS background rather than in the flat space. Therefore an important problem is to generalize the constructed currents to the AdS geometry. This problem was solved recently [19] for the case d=3.

Explicit form of the HS algebras is known for $d \leq 4$ although a conjecture was made in [18] on the structure of HS symmetries in any d. The knowledge of the structure of the HS currents in arbitrary dimension may be very useful for elucidating a structure of the HS symmetries in any d.

3   D=4 Higher Spin Algebra

The simplest global symmetry algebra of the 4d HS theory can be realized as follows. Consider the associative algebra of functions of the auxiliary Majorana spinor variables Y_ν ($\nu = 1 \div 4$) endowed with the star-product law

$$\begin{aligned}(f * g)(Y) &= \frac{1}{(2\pi)^2}\int d^4U d^4V \exp(iU_\mu V^\mu)f(Y + U)g(Y + V)\\ &= e^{i\frac{\partial}{\partial Y^1_\mu}\frac{\partial}{\partial Y^{2\mu}}}f(Y + Y^1)B(Y + Y^2)|_{Y^1=Y^2=0}.\end{aligned} \tag{33}$$

Here $f(Y)$ and $g(Y)$ are functions (polynomials or formal power series) of commuting variables Y_μ (spinor indices are raised and lowered by the 4d charge conjugation matrix $C_{\mu\nu}$: $U^\mu = C^{\mu\nu}U_\nu$, $U_\mu = U^\nu C_{\nu\mu}$). This formula defines the associative algebra with the defining relation

$Y_\mu * Y_\nu - Y_\nu * Y_\mu = 2iC_{\mu\nu}$. The star-product defined this way describes the product of Weyl ordered (i.e. totally symmetric) polynomials of oscillators in terms of symbols of operators [20]. This Weyl product law (called Moyal bracket [21] for commutators constructed from (33)) is obviously nonlocal. This is the ordinary quantum-mechanical nonlocality. Note that the integral formula (33) is in most cases more convenient for practical computations than the differential Moyal formula.

The pure Weyl algebra can only be used as the HS algebra in the bosonic case. The algebra $shsa(1)$ [22] that works also in presence of fermions is obtained by adding the "Klein operators" k and $\bar{k}$ having the properties

$$k * k = 1, \qquad \bar{k} * \bar{k} = 1, \qquad k * \bar{k} = \bar{k} * k, \tag{34}$$

$$k * y_\alpha = -y_\alpha * k, \qquad k * \bar{y}_{\dot\alpha} = \bar{y}_{\dot\alpha} * k, \qquad \bar{k} * y_\alpha = y_\alpha * \bar{k}, \qquad \bar{k} * \bar{y}_{\dot\alpha} = -\bar{y}_{\dot\alpha} * \bar{k}. \tag{35}$$

(Here $Y_\nu = (y_\alpha, \bar{y}_{\dot\alpha})$ with $\alpha = 1, 2$, $\dot\alpha = 1, 2$; in other words, moving k or $\bar{k}$ through Y_ν is equivalent to multiplication by $\pm\gamma_5$.)

The HS gauge fields are

$$W(y, \bar{y}; k, \bar{k}|x) = \sum_{A,B=0,1} \sum_{n,m=0}^{\infty} \frac{1}{2im!n!} dx^{\underline{n}} w_{\underline{n}}{}^{AB\alpha_1...\alpha_n \dot\alpha_1...\dot\alpha_m}(x) k^A \bar{k}^B y_{\alpha_1} \ldots y_{\alpha_n} \bar{y}_{\dot\alpha_1} \ldots \bar{y}_{\dot\alpha_m}. \tag{36}$$

According to [13, 23, 22] the fields $W(y, \bar{y}; k, \bar{k}|x) = W(y, \bar{y}; -k, -\bar{k}|x)$ describe the HS fields while the fields $W(y, \bar{y}; k, \bar{k}|x) = -W(y, \bar{y}; -k, -\bar{k}|x)$ are auxiliary, i.e. do not describe non-trivial degrees of freedom. Therefore we have two sets of HS potentials $w^{AA\alpha_1...\alpha_n \dot\alpha_1...\dot\alpha_m}(x)$, A=0 or 1. The subsets associated with spin s are fixed by the condition [13] $s = 1 + \frac{1}{2}(n + m)$. For a fixed value of A we therefore expect a set of currents $J^{\underline{n}}{}_{;\alpha_1...\alpha_n,\dot\alpha_1...\dot\alpha_m}(x)$. In accordance with the results of the section 2, it indeed describes in terms of two-component spinors the set of all two-row Young diagrams (7) both in the integer spin case ($n + m$ is even) and in the half-integer spin case ($n + m$ is odd), with the identification $s = \frac{1}{2}(n + m) + 1$, $t = \left[\frac{1}{2}|n - m|\right]$, where $[a]$ denotes the integer part of a. We see that HS algebras give rise to the sets of gauge fields which exactly match the sets of conserved HS currents of the section 2.

The HS curvature 2-form is

$$R(Y; K|x) = dw(Y; K|x) + (w \wedge *w)(Y; K|x), \tag{37}$$

where $d = dx^{\underline{n}} \partial_{\underline{n}}$ ($\underline{n}, \underline{m} = 0 \div 3$ are space-time (base) indices) and $K = (k, \bar{k})$ denotes the pair of the Klein operators. Let us stress that star-product acts on the auxiliary coordinates Y while the space-time coordinates $x^{\underline{n}}$ are commuting. The HS gauge transformations have a form $\delta w(Y; K|x) = d\epsilon(Y; K|x) + [w, \epsilon]_*(Y; K|x)$, where $[a, b]_* = a * b - b * a$.

A structure of HS algebras h is such that no HS ($s > 2$) field can remain massless unless it belongs to an infinite chain of fields with infinitely increasing spins. Indeed, from the definition of the star-product it follows that the gauge fields having spins s_1 and s_2 contribute to the spin $s_1 + s_2 - 2$ curvature if at least one of the spins is integer and to $s_1 + s_2 - 1$ curvature if both s_1 and s_2 are half-integer. Bilinears in the oscillators form a finite-dimensional subalgebra $sp(4|R)$

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isomorphic to AdS_4 algebra $o(3,2)$. In fact, the model has $N = 2$ SUSY [22] associated with a finite-dimensional subalgebra $osp(2,4)$.

Unbroken HS symmetries require AdS background. One can think however of some spontaneous breakdown of the HS symmetries followed by a flat contraction via a shift of the vacuum energy in the broken phase. In a physical phase with $\lambda = 0$ and $m \gg m^{exp}$ for HS fields, h should break down to a finite-dimensional subalgebra giving rise to usual lower spin gauge fields. From this perspective the Coleman-Mandula type theorems can be re-interpreted as statements concerning a possible structure of g rather than the whole HS algebra h which requires AdS geometry. These arguments are based on the $d \leq 4$ experience but we expect them to large extend to be true for higher dimensions.

4 AdS Vacuum

A structure of the full nonlinear HS equations of motion is such that any solution w_0 of the zero-curvature equation

$$dw = w * \wedge w \tag{38}$$

solves the HS equations. Such a vacuum solution has a pure gauge form

$$w_0(Y; K|x) = -g^{-1}(Y; K|x) * dg(Y; K|x) \tag{39}$$

with some invertible element $g(Y; K|x)$, i.e. $g * g^{-1} = g^{-1} * g = I$. It breaks the local HS symmetry to its stability subalgebra with the infinitesimal parameters $\epsilon_0(Y; K|x)$ satisfying the equation $D_0\epsilon_0 \equiv d\epsilon_0 - w_0 * \epsilon_0 + \epsilon_0 * w_0 = 0$ which solves as

$$\epsilon_0(Y; K|x) = g^{-1}(Y; K|x) * \epsilon_0(Y; K) * g(Y; K|x). \tag{40}$$

In the HS theories no further symmetry breaking is induced by the field equations, i.e. $\epsilon_0(Y; k, \bar{k})$ parametrizes the global symmetry of the theory. Therefore, the HS global symmetry algebra identifies with the Lie superalgebra constructed from the (anti)commutators of the elements of the Weyl algebra and its extension with the Klein operators. Note that the fields carrying odd numbers of spinor indices are anticommuting thus inducing a structure of a superalgebra into (38).

AdS background plays distinguished role in the HS theories because functions bilinear in Y_ν form a closed subalgebra with respect to commutators. This allows one to look for a solution of the vacuum equation (38) in the form

$$w_0 = \frac{1}{4i} \left(\omega_0^{\alpha\beta}(x) y_\alpha y_\beta + \bar{\omega}_0^{\dot{\alpha}\dot{\beta}}(x) \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} + \lambda h_0^{\alpha\dot{\beta}}(x) y_\alpha \bar{y}_{\dot{\beta}} \right). \tag{41}$$

Inserting these formulae into (38) one finds that the fields ω_0, $\bar{\omega}_0$ and h_0 identify with the Lorentz connection and the frame field of AdS_4, respectively, provided that the 1-form h_0 is invertible. The parameter $\lambda = r^{-1}$ is identified with the inverse AdS radius. Thus, the fact that the HS algebras are star-product (oscillator) algebras leads to the AdS geometry as a natural vacuum solution.

A particular solution of the vacuum equation (38) corresponding to the stereographic coordinates has a form

$$h_{\underline{n}}{}^{\alpha\dot\beta} = -z^{-1}\sigma_{\underline{n}}{}^{\alpha\dot\beta}\,, \tag{42}$$

$$\omega_{\underline{n}}{}^{\alpha\beta} = -\frac{\lambda^2}{2z}\left(\sigma_{\underline{n}}{}^{\alpha\dot\beta}x^\beta{}_{\dot\beta} + \sigma_{\underline{n}}{}^{\beta\dot\beta}x^\alpha{}_{\dot\beta}\right)\,, \qquad \bar\omega_{\underline{n}}{}^{\dot\alpha\dot\beta} = -\frac{\lambda^2}{2z}\left(\sigma_{\underline{n}}{}^{\alpha\dot\alpha}x_\alpha{}^{\dot\beta} + \sigma_{\underline{n}}{}^{\alpha\dot\beta}x_\alpha{}^{\dot\alpha}\right)\,, \tag{43}$$

where $\sigma_{\underline{n}}{}^{\alpha\dot\beta}$ is the set of 2×2 Hermitian matrices and we use notation

$$x^{\alpha\dot\beta} = x^{\underline{n}}\sigma_{\underline{n}}{}^{\alpha\dot\beta}\,, \quad x^2 = \frac{1}{2}x^{\alpha\dot\beta}x_{\alpha\dot\beta}\,, \quad z = 1 + \lambda^2 x^2\,. \tag{44}$$

Let us note that $z \to 1$ in the flat limit and $z \to 0$ at the boundary of AdS_4.

The form of the gauge function g reproducing these vacuum background fields (with all $s \neq 2$ fields vanishing) turns out to be remarkably simple [24]

$$g(y,\bar y|x) = 2\frac{\sqrt z}{1+\sqrt z}\exp[\frac{i\lambda}{1+\sqrt z}x^{\alpha\dot\beta}y_\alpha\bar y_{\dot\beta}] \tag{45}$$

with the inverse $g^{-1}(y,\bar y|x) = g(-y,\bar y|x)$. In many cases, g plays a role of some kind of evolution operator (cf eq. (49)). From this perspective λ is analogous to the inverse of the Planck constant, $\lambda \sim \hbar^{-1}$. This parallelism indicates that the flat limit $\lambda \to 0$ may be essentially singular.

5 Free Equations

HS symmetries mix derivatives of physical fields of all orders. To have HS symmetries linearly realized, it is useful to introduce infinite multiplets rich enough to contain dynamical fields along with all their higher derivatives. Such multiplets admit a natural realization in terms of the Weyl algebra. Namely, the 0-forms

$$C(Y|x) = C(y,\bar y|x) = \sum_{n,m=0}^\infty \frac{1}{m!n!}C^{\alpha_1\ldots\alpha_n,\,\dot\alpha_1\ldots\dot\alpha_m}(x)y_{\alpha_1}\ldots y_{\alpha_n}\bar y_{\dot\alpha_1}\ldots\bar y_{\dot\alpha_m}\,, \tag{46}$$

taking their values in the Weyl algebra form appropriate HS multiplets for lower spin matter fields and Weyl-type HS curvature tensors. The free equations of motion have a form [25]

$$\mathcal{D}_0 C \equiv dC - w_0 * C + C * \tilde w_0 = 0\,, \tag{47}$$

where tilde denotes an involutive automorphism of the HS algebra which changes a sign of the AdS translations

$$\tilde f(y,\bar y) = f(y,-\bar y)\,. \tag{48}$$

As a result, the covariant derivative $\mathcal{D}_0$ corresponds to some representation of the HS algebra which we call twisted representation. The consistency of the equation (47) is guaranteed by the vacuum equation (38). Since in this "unfolded formulation" dynamical field equations have a

form of covariant constancy conditions, one can write down a general solution of the free field equations (47) in the pure gauge form

$$C(Y|x) = g^{-1}(Y|x) * C_0(Y) * \tilde{g}(Y|x) \,, \tag{49}$$

where $C_0(Y)$ is an arbitrary $x-$independent element of the Weyl algebra.

Let us now explain in more detail a physical content of the equations (47). Fixing the AdS_4 form (41) for the vacuum background field, (47) reduces to

$$\mathcal{D}_0 C(y, \bar{y}|x) \equiv D^L C(y, \bar{y}|x) + \lambda\{h^{\alpha\dot{\beta}} y_\alpha \bar{y}_{\dot{\beta}}, C(y, \bar{y}|x)\}_* = 0 \,, \tag{50}$$

where $\{a, b\}_* = a * b + b * a$ and

$$D^L C(y, \bar{y}|x) = dC(y, \bar{y}|x) + \frac{i}{4}([\omega^{\alpha\beta} y_\alpha y_\beta + \bar{\omega}^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}}, C(y, \bar{y}|x)]_*) \,. \tag{51}$$

Inserting (46) one arrives at the following infinite chain of equations

$$D^L C_{\alpha(m), \dot{\beta}(n)} = i\lambda h^{\gamma\dot{\delta}} C_{\alpha(m)\gamma, \dot{\beta}(n)\dot{\delta}} - inm\lambda h_{\alpha\dot{\beta}} C_{\alpha(m-1), \dot{\beta}(n-1)} \,, \tag{52}$$

where D^L is the Lorentz-covariant differential $D^L A_{\alpha\dot{\beta}} = dA_{\alpha\dot{\beta}} + \omega_\alpha{}^\gamma \wedge A_{\gamma\dot{\beta}} + \bar{\omega}_{\dot{\beta}}{}^{\dot{\delta}} \wedge A_{\alpha\dot{\delta}}$. Here we skip the subscript 0 referring to the vacuum AdS solution and use again the convention with symmetrized indices denoted by the same letter and a number of symmetrized indices indicated in brackets. The system (52) decomposes into a set of independent subsystems with $n - m$ fixed. It turns out [25] that the subsystem with $|n - m| = 2s$ describes a massless field of spin s (note that the fields $C_{\alpha(m), \dot{\beta}(n)}$ and $C_{\beta(n), \dot{\alpha}(m)}$ are complex conjugated).

For example, the sector of $s = 0$ is associated with the fields $C_{\alpha(n), \dot{\beta}(n)}$. The equation (52) at $n = m = 0$ expresses the field $C_{\alpha, \dot{\beta}}$ via the first derivative of C as $C_{\alpha, \dot{\beta}} = \frac{1}{2i\lambda} h^n_{\alpha\dot{\beta}} D^L_{\underline{n}} C$, where $h^n_{\alpha\dot{\beta}}$ is the inverse frame field ($h^n_{\alpha\dot{\beta}} h^{\gamma\dot{\delta}}_{\underline{n}} = 2\delta^\gamma_\alpha \delta^{\dot{\delta}}_{\dot{\beta}}$ with the normalization chosen to be true for $h^n_{\alpha\dot{\beta}} = \sigma^n_{\alpha\dot{\beta}}$ and $h^{\alpha\dot{\beta}}_{\underline{n}} = \sigma^{\alpha\dot{\beta}}_{\underline{n}}$). The second equation with $n = m = 1$ contains more information. First, one obtains by contracting indices with the frame field that $h^n_{\alpha\dot{\beta}}(D^L_{\underline{n}} C^{\alpha, \dot{\beta}} + 8i\lambda C) = 0$. This reduces to the Klein-Gordon equation in AdS_4

$$\Box C - 8\lambda^2 C = 0 \,. \tag{53}$$

The rest part of the equation (52) with $n = m = 1$ expresses the field $C_{\alpha\alpha, \dot{\beta}\dot{\beta}}$ via second derivatives of C: $C_{\alpha\alpha, \dot{\beta}\dot{\beta}} = \frac{1}{(2i\lambda)^2} h^n_{\alpha\dot{\beta}} D^L_{\underline{n}} h^m_{\alpha\dot{\beta}} D^L_{\underline{m}} C$. All other equations with $n = m > 1$ either reduce to identities by virtue of the spin 0 dynamical equation (53) or express higher components in the chain of fields $C_{\alpha_1 \ldots \alpha_n, \dot{\beta}_1 \ldots \dot{\beta}_n}$ via higher derivatives in the space-time coordinates. The value of the mass parameter in (53) is such that C describes a massless scalar in AdS_4.

For spins $s \geq 1$ it is more useful to treat the equations (52) not as fundamental ones but as consequences of the HS equations formulated in terms of gauge fields (potentials). To illustrate this point let us consider the example of gravity. As argued in section 3, Lorentz connection

1−forms $\omega_{\alpha\beta}$, $\bar{\omega}_{\dot{\alpha}\dot{\beta}}$ and vierbein 1−form $h_{\alpha\dot{\beta}}$ can be identified with the $sp(4)$−gauge fields. The $sp(4)$−curvatures read

$$R_{\alpha_1\alpha_2} = d\omega_{\alpha_1\alpha_2} + \omega_{\alpha_1}{}^{\gamma} \wedge \omega_{\alpha_2\gamma} + \lambda^2\, h_{\alpha_1}{}^{\dot{\delta}} \wedge h_{\alpha_2\dot{\delta}}\,, \tag{54}$$

$$\bar{R}_{\dot{\alpha}_1\dot{\alpha}_2} = d\bar{\omega}_{\dot{\alpha}_1\dot{\alpha}_2} + \bar{\omega}_{\dot{\alpha}_1}{}^{\dot{\gamma}} \wedge \bar{\omega}_{\dot{\alpha}_2\dot{\gamma}} + \lambda^2\, h^{\gamma}{}_{\dot{\alpha}_1} \wedge h_{\gamma\dot{\alpha}_2}\,, \tag{55}$$

$$r_{\alpha\dot{\beta}} = dh_{\alpha\dot{\beta}} + \omega_{\alpha}{}^{\gamma} \wedge h_{\gamma\dot{\beta}} + \bar{\omega}_{\dot{\beta}}{}^{\dot{\delta}} \wedge h_{\alpha\dot{\delta}}\,. \tag{56}$$

The zero-torsion condition $r_{\alpha\dot{\beta}} = 0$ expresses the Lorentz connection ω and $\bar{\omega}$ via derivatives of h. After that, the λ−independent part of the curvature 2−forms R (54) and $\bar{R}$ (55) coincides with the Riemann tensor. Einstein equations imply that the Ricci tensor vanishes up to a constant trace part proportional to the cosmological constant. This is equivalent to saying that only those components of the tensors (54) and (55) are allowed to be non-zero which belong to the Weyl tensor. The Weyl tensor is described by the fourth-rank mutually conjugated totally symmetric multispinors $C_{\alpha_1\alpha_2\alpha_3\alpha_4}$ and $\bar{C}_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}$. Therefore, Einstein equations with the cosmological term can be cast into the form

$$r_{\alpha\dot{\beta}} = 0\,, \tag{57}$$

$$R_{\alpha_1\alpha_2} = h^{\gamma_1\dot{\delta}} \wedge h^{\gamma_2}{}_{\dot{\delta}} C_{\alpha_1\alpha_2\gamma_1\gamma_2}\,, \qquad \bar{R}_{\dot{\beta}_1\dot{\beta}_2} = h^{\eta\dot{\delta}_1} \wedge h_{\eta}{}^{\dot{\delta}_2} \bar{C}_{\dot{\beta}_1\dot{\beta}_2\dot{\delta}_1\dot{\delta}_2}\,. \tag{58}$$

It is useful to treat the 0−forms $C_{\alpha(4)}$ and $\bar{C}_{\dot{\alpha}(4)}$ as independent field variables which identify with the Weyl tensor by virtue of the equations (58). From (58) it follows that the 0−forms $C_{\alpha(4)}$ and $\bar{C}_{\dot{\alpha}(4)}$ should obey certain differential restrictions as a consequence of the Bianchi identities for the curvatures R and $\bar{R}$. It is not difficult to make sure that these differential restrictions just have the form of the equations (52) with $n = 4$, $m = 0$ and $n = 0$, $m = 4$ with the fields $C_{\alpha(5),\dot{\delta}}$ and $\bar{C}_{\gamma,\dot{\beta}(5)}$ describing unrestricted components of the first derivatives of the Weyl tensor. The consistency conditions for these relations are expressed by the equations (52) with $n = 5$, $m = 1$ and $n = 1$, $m = 5$. Continuation of this process leads in the linearized approximation to the infinite chains of differential relations (52) with $|n - m| = 4$. All these relations contain no new dynamical information compared to that contained in the original Einstein equations in the form (57), (58), merely expressing the highest 0−forms $C_{\alpha(n+4),\dot{\beta}(n)}$ and $\bar{C}_{\alpha(n),\dot{\beta}(n+4)}$ via derivatives of the lowest 0−forms $C_{\alpha(4)}$ and $\bar{C}_{\dot{\beta}(4)}$. As a result, the system of equations obtained in such a way turns out to be equivalent to the Einstein equations with the cosmological term.

As shown in [13, 25] this construction extends to all spins $s \geq 1$. In terms of the linearized HS curvatures

$$R_1(y,\bar{y} \mid x) \equiv dw(y,\bar{y} \mid x) - w_0(y,\bar{y} \mid x) * w(y,\bar{y} \mid x) + w(y,\bar{y} \mid x) * w_0(y,\bar{y} \mid x) \tag{59}$$

the linearized HS equations read

$$R_1(y,\bar{y}|x) = h^{\gamma\dot{\beta}} \wedge h_{\gamma}{}^{\dot{\alpha}} \frac{\partial}{\partial\bar{y}^{\dot{\alpha}}} \frac{\partial}{\partial\bar{y}^{\dot{\beta}}} C(0,\bar{y}|x) + h^{\alpha\dot{\gamma}} \wedge h^{\beta}{}_{\dot{\gamma}} \frac{\partial}{\partial y^{\alpha}} \frac{\partial}{\partial y^{\beta}} C(y,0|x) \tag{60}$$

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together with (50). This statement, which plays a key role from various points of view, we call Central On-Mass-Shell Theorem. Eqs. (60) and (50) contain [13] the usual free HS equations in AdS_4, which follow from the standard actions proposed in [6], and express all auxiliary components via higher derivatives of the dynamical fields

$$C_{\alpha(n),\dot{\beta}(m)} = \frac{1}{(2i\lambda)^{\frac{1}{2}(n+m-2s)}} h^{\underline{n}_1}_{\alpha\dot{\beta}} D^L_{\underline{n}_1} \ldots h^{\underline{n}_{\frac{1}{2}(n+m-2s)}}_{\alpha\dot{\beta}} D^L_{\underline{n}_{\frac{1}{2}(n+m-2s)}} C_{\alpha(2s)} \quad n \geq m \qquad (61)$$

(and complex conjugated) for the 0-forms and by analogous formulae for the gauge 1-forms [13]. A spin $s \geq 1$ dynamical massless field is identified with the 1-form (potential) $w_{\alpha(s-1),\dot{\beta}(s-1)}$ for integer $s \geq 1$ or $w_{\alpha(s-3/2),\dot{\beta}(s-1/2)}$ and $w_{\alpha(s-1/2),\dot{\beta}(s-3/2)}$ for half-integer $s \geq 3/2$. The matter fields are described by the 0-forms $C_{\alpha(0),\dot{\alpha}(0)}$ for $s = 0$ or $C_{\alpha(1),\dot{\alpha}(0)}$ and $C_{\alpha(0),\dot{\alpha}(1)}$ for $s = 1/2$. The infinite set of the $0-$forms C forms a basis in the space of all on-mass-shell nontrivial combinations of the covariant derivatives of the matter fields and (HS) curvatures.

Note that the relationships (60) and (50) link derivatives in the space-time coordinates $x^{\underline{n}}$ with those in the auxiliary spinor variables y_α and $\bar{y}_{\dot{\alpha}}$. In accordance with (61), in the sector of 0-forms the derivatives in the auxiliary spinor variables can be viewed as a square root of the space-time derivatives,

$$\frac{\partial}{\partial x^{\underline{n}}} C(y,\bar{y}|x) \sim \lambda h_{\underline{n}}^{\alpha\dot{\beta}} \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \bar{y}^{\dot{\beta}}} C(y,\bar{y}|x) . \qquad (62)$$

As a result, the nonlocality of the star-product (33) acting on the auxiliary spinor variables indicates a potential nonlocality in the space-time sense. The HS equations contain star-products via terms $C(Y|x) * X(Y|x)$ with some operators X constructed from the gauge and matter fields. Once $X(Y|x)$ is at most quadratic in the auxiliary variables Y^ν, the resulting expressions are local, containing at most two derivatives in Y^ν. This is the case for the AdS background gravitational fields and therefore, in agreement with the analysis of this section, the HS dynamics is local at the linearized level. But this may easily be not the case beyond the linearized approximation. Another important consequence of the formula (62) is that it contains explicitly the inverse AdS radius λ and becomes meaningless in the flat limit $\lambda \to 0$. This happens because, when resolving these equations for the derivatives in the auxiliary variables y and $\bar{y}$, the space-time derivatives appear in the combination $\lambda^{-1}\frac{\partial}{\partial x^{\underline{n}}}$ that leads to the inverse powers of λ in front of the terms with higher derivatives in the HS interactions. This is the main reason why HS interactions require the cosmological constant to be nonzero as was first concluded in [10]. To summarize, the following facts are strongly correlated:
(i) HS algebras are described by the (Moyal) star-product in the auxiliary spinor space;
(ii) relevance of the AdS background;
(iii) potential space-time nonlocality of the HS interactions due to the appearance of higher derivatives at the nonlinear level.

These properties are in many respects reminiscent of the superstring picture with the parallelism between the cosmological constant and the string tension parameter. The fact that unbroken HS symmetries require AdS geometry may provide an explanation why the symmetric HS phase is not visible in the usual superstring picture with the flat background space-time.

The fact that $C(Y|x)$ describes all derivatives of the physical fields compatible with the field equations allows us to solve the dynamical equations in the form (49). The arbitrary parameters $C_0(Y)$ in (49) describe all higher derivatives of the field $C(Y|x_0)$ at the point x_0 with $g(Y|x_0) = I$ ($x_0 = 0$ for the gauge function (45)). In other words, (49) describes a covariantized Taylor expansion in some neighborhood of x_0. To illustrate how the formula (49) can be used to produce explicit solutions of the HS equations in AdS_4 let us set

$$C_0(Y) = \exp i(y^\alpha \eta_\alpha + \bar{y}^{\dot\alpha} \bar{\eta}_{\dot\alpha}) \,, \tag{63}$$

where η_α is an arbitrary commuting complex spinor and $\bar{\eta}_{\dot\alpha}$ is its complex conjugate. Taking into account that $\tilde{g}^{-1}(Y|x) = g(Y|x)$, inserting $g(Y|x)$ into (49) and using the product law (33) one performs elementary Gaussian integrations to obtain [24]

$$C(Y|x) = z^2 \exp i \left[-\lambda(y_\alpha \bar{y}_{\dot\beta} + \eta_\alpha \bar{\eta}_{\dot\beta}) x^{\alpha\dot\beta} + z(y^\alpha \eta_\alpha + \bar{y}^{\dot\alpha} \bar{\eta}_{\dot\alpha}) \right] \,, \tag{64}$$

where $z = 1 + \lambda^2 \frac{1}{2} x^{\alpha\dot\beta} x_{\alpha\dot\beta}$. From $C_{\alpha_1\ldots\alpha_n}(x) = \frac{\partial}{\partial y^{\alpha_1}} \cdots \frac{\partial}{\partial y^{\alpha_n}} C(y, \bar{y}|x)|_{y=\bar{y}=0}$, it follows then for the matter fields and HS Weyl tensors

$$C_{\alpha\ldots\alpha_{2s}}(x) = z^{2(s+1)} \eta_{\alpha_1} \ldots \eta_{\alpha_{2s}} \exp i k_{\gamma\dot\beta} x^{\gamma\dot\beta} \,, \tag{65}$$

where $k_{\alpha\dot\beta} = -\lambda \eta_\alpha \bar{\eta}_{\dot\beta}$ is a null vector expressed in the standard way in terms of commuting spinors. (Expressions for the conjugated Weyl tensors carrying dotted indices are analogous). Since $z \to 1$ in the flat limit, the obtained solution describes plane waves in the flat limit $\lambda \to 0$ provided that the parameters η_α and $\bar{\eta}_{\dot\alpha}$ are rescaled according to $\eta_\alpha \to \lambda^{-1/2} \tilde{\eta}_\alpha$, $\bar{\eta}_{\dot\alpha} \to \lambda^{-1/2} \tilde{\bar{\eta}}_{\dot\alpha}$. On the other hand, $z \to 0$ at the boundary of AdS_4 and therefore the constructed AdS plane waves tend to zero at the boundary.

Let us note that although the equation (60) does not have a form of a zero-curvature equation, it also can be solved explicitly [24] using a more sophisticated technics inspired by the analysis of the nonlinear HS dynamics.

6   Nonlinear Higher Spin Equations

Now we discuss the full nonlinear system of 4d HS equations following [11, 26, 27]. The resulting formulation, amounts to certain non-commutative Yang-Mills fields and is interesting on its own right.

The key element of the construction consists of the doubling of auxiliary Majorana spinor variables Y_ν in the HS 1-forms $w(Y; K|x) \longrightarrow W(Z; Y; K|x)$ and 0-forms $C(Y; K|x) \longrightarrow B(Z; Y; K|x)$. The dependence on the additional variables Z_ν is determined in terms of "initial data"

$$w(Y; K|x) = W(0; Y; K|x) \,, \qquad C(Y; K|x) = B(0; Y; K|x). \tag{66}$$

by appropriate equations and effectively describes all nonlinear corrections to the field equations. To this end we introduce a compensator-type spinor field $S_\nu(Z; Y; K|x)$ which does not carry its own degrees of freedom and plays a role of a covariant differential along the additional

Z_ν directions. It is convenient to introduce anticommuting $Z-$differentials $dZ^\nu dZ^\mu = -dZ^\mu dZ^\nu$ to interpret $S_\nu(Z;Y;K|x)$ as a Z 1-form $S = dZ^\nu S_\nu$.

The nonlinear HS dynamics is formulated in terms of the star-product

$$(f * g)(Z;Y) = \frac{1}{(2\pi)^4} \int d^4U\, d^4V \exp\left[iU^\mu V^\nu C_{\mu\nu}\right] f(Z+U;Y+U)g(Z-V;Y+V), \quad (67)$$

where U^μ and V^μ are real integration variables. It is a simple exercise with Gaussian integrals to see that this star-product is associative $f * (g * h) = (f * g) * h$ and is normalized such that 1 is a unit element of the star-product algebra, i.e. $f * 1 = 1 * f = f$. The star-product (67) again yields a particular realization of the Weyl algebra[1]. The following simple formulae are true

$$[Y_\mu, f]_* = 2i\frac{\partial f}{\partial Y^\mu}, \qquad [Z_\mu, f]_* = -2i\frac{\partial f}{\partial Z^\mu}, \quad (68)$$

for any $f(Z,Y)$. From (67) it follows that functions $f(Y)$ independent of Z form a proper subalgebra with the Weyl star-product (33). An important property of the star-product (67) is that it admits the inner Klein operators $\upsilon = \exp iz_\alpha y^\alpha$ and $\bar\upsilon = \exp i\bar z_{\dot\alpha} \bar y^{\dot\alpha}$, having the properties $\upsilon * \upsilon = \bar\upsilon * \bar\upsilon = 1$ and

$$\upsilon * f(z, \bar z; y, \bar y) = f(-z, \bar z; -y, \bar y) * \upsilon, \quad \bar\upsilon * f(z, \bar z; y, \bar y) = f(z, -\bar z; y, -\bar y) * \bar\upsilon. \quad (69)$$

The star-product (67) is regular: given two polynomials f and g, $f * g$ is also some polynomial. The special property of the star-product (67) is that it is defined for the class of nonpolynomial functions [28, 29] which appear in the process of solution of the nonlinear HS equations and contains the Klein operators υ and $\bar\upsilon$.

The full system of 4d equations has the form

$$dW = W * W, \qquad dB = W * B - B * W, \qquad dS = W * S - S * W, \quad (70)$$

$$S * B = B * S, \qquad S * S = dZ^\nu dZ^\mu \left(-iC_{\nu\mu} + 4R_{\nu\mu}(B)\right), \quad (71)$$

where $C_{\nu\mu}$ is the charge conjugation matrix and $R_{\nu\mu}(B)$ is a certain star-product function of the field B and some central elements of the algebra. The function $R_{\nu\mu}(B)$ that encodes all information about the HS dynamics has the form

$$dZ^\nu dZ^\mu R_{\nu\mu}(B) = \frac{1}{4i}\left(dz_\alpha dz^\alpha \left(\nu + \eta\mathcal{F}(B)\right) * k * \upsilon + d\bar z_{\dot\alpha}\, d\bar z^{\dot\alpha} \left(\bar\nu + \bar\eta\bar{\mathcal{F}}(B)\right) * \bar k * \bar\upsilon\right). \quad (72)$$

Here ν, $\bar\nu$, η and $\bar\eta$ are arbitrary parameters. The parameters η and $\bar\eta$ play a role of the coupling constants while the auxiliary parameters ν and $\bar\nu$ are introduced for the future convenience and can be set equal to zero at least for the most symmetric vacuum solution. The function $\mathcal{F}$ describes the ambiguity in the HS interactions. The simplest choice $\mathcal{F}(B) = B$ leads to the nontrivial (nonlinear) dynamics. The case with $\nu = \bar\nu = \mathcal{F} = 0$ leads to the free field equations.

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[1]The star-product (67) corresponds to the normal ordering of the Weyl algebra with respect to the creation and annihilation operators $a_\mu^+ = \frac{1}{2}(Y_\mu - Z_\mu)$ and $a_\mu = \frac{1}{2}(Y_\mu + Z_\mu)$ satisfying the commutation relations $[a_\mu, a_\nu]_* = [a_\mu^+, a_\nu^+]_* = 0$, $[a_\mu, a_\nu^+]_* = iC_{\mu\nu}$.

Note that the exterior Klein operator k $(\bar{k})$ anticommutes with all left (right) spinors including the differentials dz^α $(d\bar{z}^{\dot\alpha})$.

The equations (70) and (71) are invariant under the gauge transformations

$$\delta W = d\varepsilon + [\varepsilon, W]_* , \qquad \delta S = [\varepsilon, S]_* , \qquad \delta B = [\varepsilon, B]_* . \tag{73}$$

The space-time differential d only emerges in the equations (70) which have a form of zero-curvature and covariant constancy conditions and therefore admit explicit solution in the pure gauge form analogous to (39) and (40)

$$W = -g^{-1}(Z;Y;K|x) * dg(Z;Y;K|x) , \tag{74}$$

$$B(Z;Y;K|x) = g^{-1}(Z;Y;K|x) * b(Z;Y;K) * g(Z;Y;K|x) , \tag{75}$$

$$S(Z;Y;K|x) = g^{-1}(Z;Y;K|x) * s(Z;Y;K) * g(Z;Y;K|x) \tag{76}$$

with some invertible $g(Z;Y;K|x)$ and arbitrary $x-$independent functions $b(Z;Y;K)$ and $s(Z;Y;K)$. Due to the gauge invariance of the whole system one is left only with the equations (71) for $b(Z;Y;Q)$ and $s(Z;Y;Q)$. These encode in a coordinate independent way all information about the dynamics of massless fields of all spins. In fact, the "constraints" (71) just impose appropriate restrictions on the quantities b and s to guarantee that the original space-time equations of motion are satisfied.

In the analysis of the HS dynamics, a typical vacuum solution for the field S is $S_0 = dZ^\nu Z_\nu$. From (68) it follows that

$$[S_0, f]_* = -2i\partial f , \qquad \partial = dZ^\nu \frac{\partial}{\partial Z^\nu} . \tag{77}$$

Interpreting the deviation of the full field S from the vacuum value S_0 as a $Z-$component of the gauge field W,

$$S = S_0 + 2i dZ^\nu W_\nu , \tag{78}$$

one rewrites the equations (70), (71) as

$$\mathcal{R} = dZ^\nu dZ^\mu \, R_{\nu\mu}(B) , \qquad \mathcal{D}B = 0 , \tag{79}$$

where the generalized curvatures and covariant derivative are defined by the relations

$$\mathcal{R} = (d + \partial)(dx^{\underline{n}}W_{\underline{n}} + dZ^\nu W_\nu) - (dx^{\underline{n}}W_{\underline{n}} + dZ^\nu W_\nu) \wedge (dx^{\underline{n}}W_{\underline{n}} + dZ^\nu W_\nu) , \tag{80}$$

$$\mathcal{D}(A) = (d + \partial)A - (dx^{\underline{n}}W_{\underline{n}} + dZ^\nu W_\nu) * A + A * (dx^{\underline{n}}W_{\underline{n}} + dZ^\nu W_\nu) . \tag{81}$$

$(dx^{\underline{n}}dZ^\nu = -dZ^\nu dx^{\underline{n}}.)$ We see that the function $R_{\nu\mu}(B)$ in (71) identifies with the ZZ components of the generalized curvatures, while xx and xZ components of the curvature vanish. The equation $\mathcal{D}B = 0$ means that the curvature $R_{\nu\mu}(B)$ is covariantly constant. In fact, it is the compatibility condition for the equations (80) and (72).

The consistency of the system of equations (70), (71) guarantees that it admits a perturbative solution as a system of differential equations with respect to Z_ν. A natural vacuum solution

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is $W_0(Z; Y; K|x) = w_0(Y|x)$, $B_0 = 0$ and $S_{0\nu} = Z_\nu$ ($\nu = \bar{\nu} = 0$) with the field w_0 (41) describing the AdS_4 vacuum. All fluctuations of the fields can be expressed modulo gauge transformations in terms of the "initial data" (66) identified with the physical HS fields. Inserting thus obtained expressions into (70) one reconstructs all nonlinear corrections to the free field equations.

In our approach, non-commutative gauge fields appear in the auxiliary spinor space associated with the coordinates Z^ν. The dynamics of the HS gauge fields is formulated entirely in terms of the corresponding non-commutative gauge curvatures. For the first sight, it is very different from the non-commutative Yang-Mills model considered recently in [5] in the context of the new phase of string theory, in which star-product is defined directly in terms of the original space-time coordinates $x^{\underline{n}}$. However, the difference may be not that significant taking into account the relationships like (62) between space-time and spinor derivatives, which are themselves consequences of the equations (70). From this perspective, the situation with the HS equations is reminiscent of the approach developed in [30] to solve the problem of quantization of symplectic structures in which the complicated problem of quantization of some (base) manifold (coordinates $x^{\underline{n}}$) is reduced to a simpler problem of quantization in the fibre endowed with the Weyl star-product structure (analog of coordinates Z^n). The difference between the Fedosov's approach and the structures underlying the HS equations is that the former is based on the vector fiber coordinates Z^n, while the HS dynamics chooses spinor coordinates Z^ν. The same difference is obvious in the context of a possible relationship of the HS theories with M theory. However, such a relationship should be reconsidered in presence of non-zero vacuum antisymmetric tensor fields. Antisymetric tensor fields are indeed present in the HS theories in the sector of auxiliary fields B containing even combinations of the Klein operators. Most likely, the corresponding vacuum solution of the HS equations will be non-polynomial in the spinor variables Y_ν in the sector of HS gauge 1-forms. If so, the resulting HS equations will become space-time non-local even at the free field level. An interesting problem for the future is therefore to investigate the explicit character of this non-locality in presence of the non-zero antisymmetric tensor fields to check whether or not it develops the non-commutative structure in the space-time sense. Let us note that a relationship between non-commutative Yang-Mills theory and Fedosov approach has been discussed in the recent paper [31].

An important property of the d4 equations is the existence of the flows with respect of the coupling constants η and $\bar{\eta}$ that commute to the whole system (70), (71) and to each other [26, 27],

$$\frac{\partial X}{\partial \eta} = \frac{\partial X}{\partial \nu} * \mathcal{F}(B), \qquad \frac{\partial X}{\partial \bar{\eta}} = \frac{\partial X}{\partial \bar{\nu}} * \bar{\mathcal{F}}(B) \tag{82}$$

for $X = W$, S or B (other forms of the flows suggested in [27] are equivalent to (82) modulo gauge transformations). The integrating flows (82) manifest the simple fact that B behaves like a constant in the system (70)-(71): it commutes to S_ν and satisfies the covariant constancy condition. Indeed, the meaning of (82) is that the derivative with respect to $\eta \mathcal{F}(B)$ is the same as that with respect to ν. The parameter η can be identified with the coupling constant. The flows (82) therefore describe the evolution with respect to the coupling constant.

The flows (82) reduce the problem of solving the nonlinear HS equations to the ordinary differential equations with respect to η and $\bar{\eta}$ provided that the "initial data" problem with $B = 0$ is solved for arbitrary constants ν and $\bar{\nu}$. Remarkably, the latter problem admits explicit

solution [29]. We believe that this indicates some sort of integrability of the full nonlinear system of 4d HS equations. Such an approach is very efficient at least perturbatively allowing one to solve equations by iterating the flows (82). In particular this technics was used in [24] to reconstruct the form of plane wave AdS_4 HS potentials in terms of the field strengths.

Although the mapping induced by the flow (82) does not manifestly contain space-time derivatives, it contains them implicitly via highest components $C_{\nu(n)}$ of the generating function which are identified with the highest derivatives of the dynamical fields according to (61). For example, the equation (82) in the zero order in η reads in the sector of B

$$\frac{\partial}{\partial \eta} B_1(Z, Y) = \frac{\partial C(Z, Y)}{\partial \nu} * C(Y) . \tag{83}$$

Because of the nonlocality of the star-product, for each fixed rank multispinorial component of the left hand side of this formula there appears, in general, an infinite series involving bilinear combinations of the components $C_{\nu(n)}$ with all n on the right hand side of (83). Therefore, the right hand side of (83) effectively involves space-time derivatives of all orders, i.e. may describe some nonlocal transformation. Therefore, the system (70), (71) cannot be treated as locally equivalent to the free system ($\eta = \bar{\eta} = 0$). Instead we can only claim that there exists a nonlocal mapping between the free and nonlinear system.

For the first sight the existence of the 4d flows is paradoxical because it establishes a connection between the full nonlinear problem and the free system with vacuum fields in the sector of the gauge fields W. For example, in the gravity sector, the appropriate version of the Einstein equations has the form (57), (58) with the Weyl tensor components $C_{\alpha(4)}$ and $C_{\dot{\alpha}(4)}$ replaced by $\eta C_{\alpha(4)}$ and $\bar{\eta} C_{\dot{\alpha}(4)}$, respectively. In the limit $\eta = 0$, Einstein equations therefore reduce to the vacuum equations of the AdS space. The dynamical equations of the massless spin 2 field reappear as equations on the Weyl tensor contained in (50). What happens is that the integrating flow generates a sort of a normal coordinate expansion of the form $W = W_0 + \eta \alpha_1(x) C + \eta^2 \alpha_2(x) C^2 + \ldots$, providing a systematic way for the derivation of the coefficients of the expansions in powers of (HS) Weyl tensors C. The procedure is purely algebraic at any given order in η (equivalently C). In particular, such an expansion reconstructs the metric tensor in terms of the curvature tensor.

7   Conclusions

HS gauge theories are based on the infinite-dimensional HS symmetries realized as the algebras of oscillators carrying spinorial representations of the space-time symmetries [33]. These star-product algebras exhibit the usual quantum-mechanical nonlocality in the auxiliary spinor spaces. An important point is that the dynamical HS field equations transform this nonlocality into space-time nonlocality, i.e. the quantum mechanical nonlocality of the HS algebras may imply some space-time nonlocality of the HS gauge theories at the interaction level. The same time the HS gauge theories remain local at the linearized level. The relevant geometric setting is provided by the Weyl bundles with space-time base manifold and Weyl algebras with spinor generating elements as the fibre. The star-product acts in the fiber rather than directly in the space-time. The noncommutative Yang-Mills theory structure also appears in the fiber sector.

An important implication of the star-product origin of the HS algebras is that the space-time symmetries are simple and therefore correspond to AdS geometry rather than to the flat one. The space-time symmetries are realized in terms of bilinears in spinor oscillators according to the isomorphism $o(3,2) \sim sp(4;R)$. This phenomenon has two consequences. On the one hand, it explains why the theory is local at the linearized level. The reason is that bilinears in the non-commuting auxiliary coordinates can lead to at most two derivatives in the star-products. On the other hand, the fact that HS models require AdS geometry is closely related to their potential nonlocality at the interaction level because it allows expansions with arbitrary high space-time derivatives, in which the coefficients carry appropriate (positive or negative) powers of the cosmological constant fixed by counting of dimensions. As a result, HS symmetries link together such seemingly distinct concepts as AdS geometry, space-time nonlocality of interactions and quantum mechanical nonlocality of the star-products in auxiliary spinor spaces.

Another consequence of the star-product origin of the HS symmetries is that HS theories are based on the associative structure rather than on the Lie-algebraic one. As a result, the construction can be extended [25] to the case with inner symmetries by endowing all fields with the matrix indices. HS gauge theories with non-Abelian symmetries classify [32] in a way analogous to the Chan-Paton symmetries in oriented and non-oriented strings. Some of them exhibit N-extended space-time supersymmetries [33, 34].

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